Case Study:
Dense Matrix-Vector Multiplication

Dense matrix-vector multiplication in DP

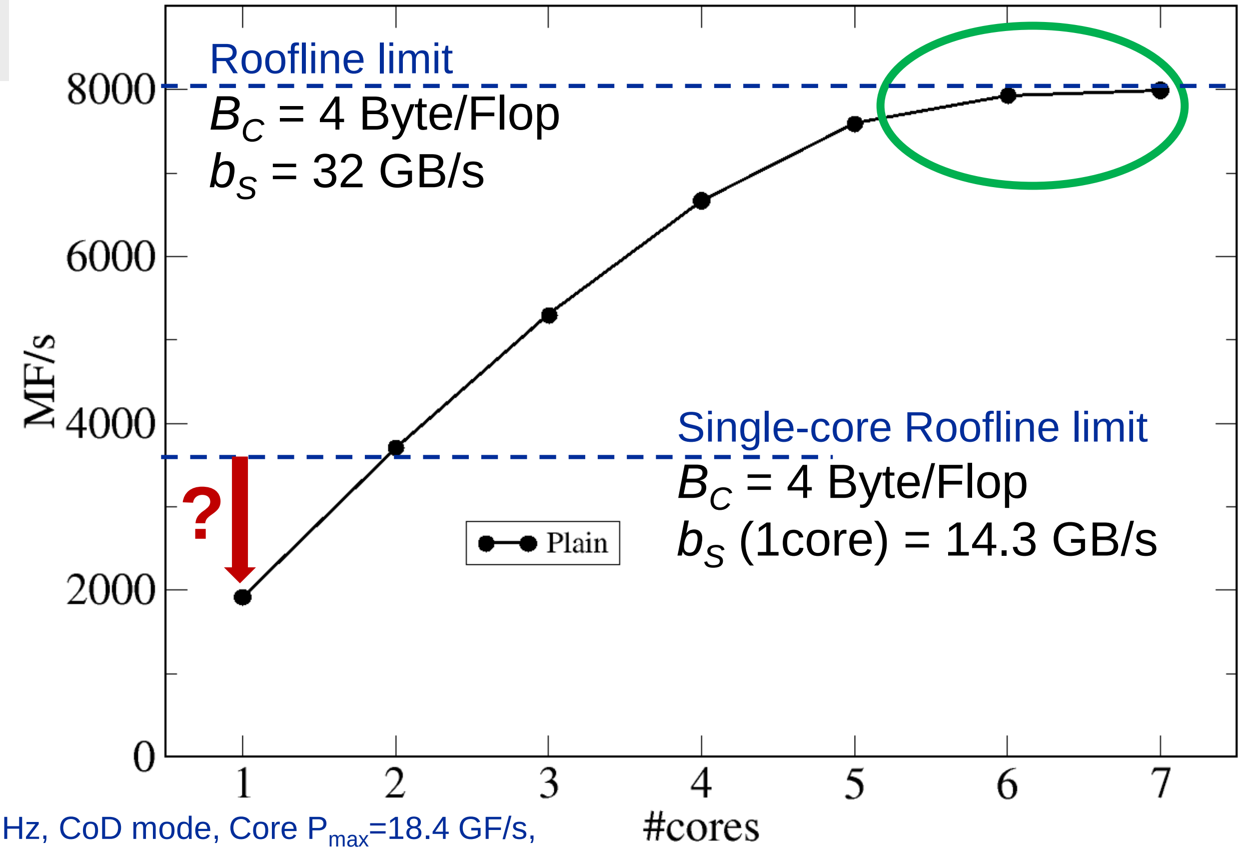
```
do c = 1 , NC
 do r = 1 , NR
     y(r)=y(r) + A(r,c)* x(c)
 enddo
enddo
```

```
do c = 1 , NC
  tmp=x(c)
  do r = 1 , NR
     y(r)=y(r) + A(r,c)* tmp
  enddo
enddo
```

dMVM scaling w/ OpenMP

```
!$omp parallel do reduction(+:y)
do c = 1 , NC
  do r = 1 , NR
     y(r) = y(r) + A(r,c) * x(c)
enddo ; enddo
!$omp end parallel do
```

NR=40,000; NC=10,000



Intel Xeon E5 2695 v3 (Haswell-EP), 2.3 GHz, CoD mode, Core P_{max}=18.4 GF/s,
Caches: 32 KB / 256 KB / 35 MB, PageSize: 2 MB; ifort V15.0.1.133; b_S = 32 Gbyte/s

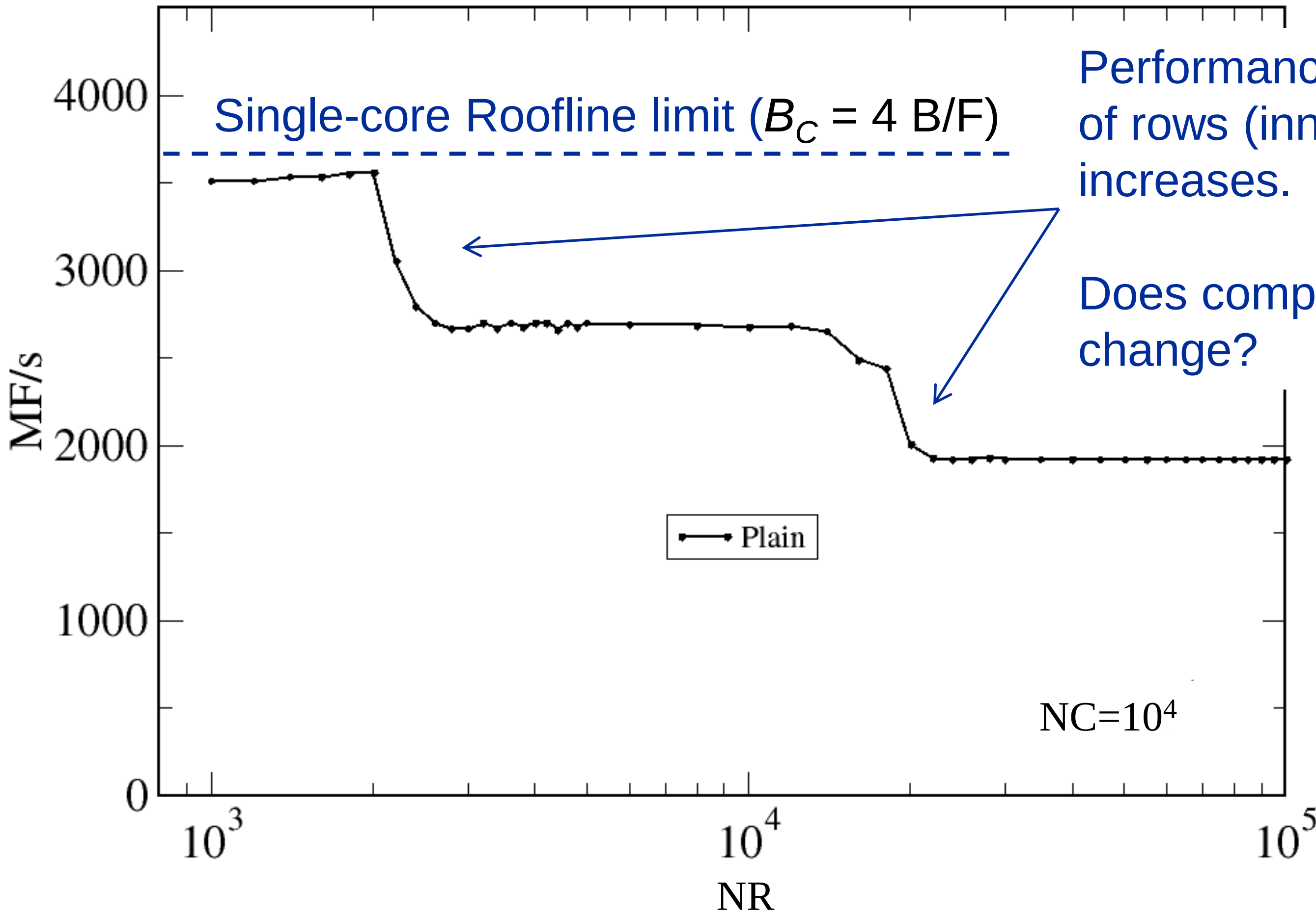
DMVM (DP) – Reminder on AVX vectorization

- Vectorization strategy: 4-way inner loop unrolling
- One register holds `tmp` in each of its 4 entries ("broadcast")

```
do c = 1,NC
  tmp=x(c)
  do r = 1,NR,4   ! R is multiple of 4
    y(r)   = y(r)   + A(r,c)  * tmp
    y(r+1) = y(r+1) + A(r+1,c)* tmp
    y(r+2) = y(r+2) + A(r+2,c)* tmp
    y(r+3) = y(r+3) + A(r+3,c)* tmp
  enddo
enddo
```

- Loop kernel requires/consumes 3 AVX registers
- Extra 3-way unrolling required to overcome ADD pipeline stalls

DMVM (DP) – Single core performance vs. column height



Performance drops as number of rows (inner loop length) increases.

Does computational intensity change?

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Caches: 32 KB / 256 KB / 35 MB, PageSize: 2 MB; ifort V15.0.1.133; b_S = 32 Gbyte/s

DMVM data traffic analysis

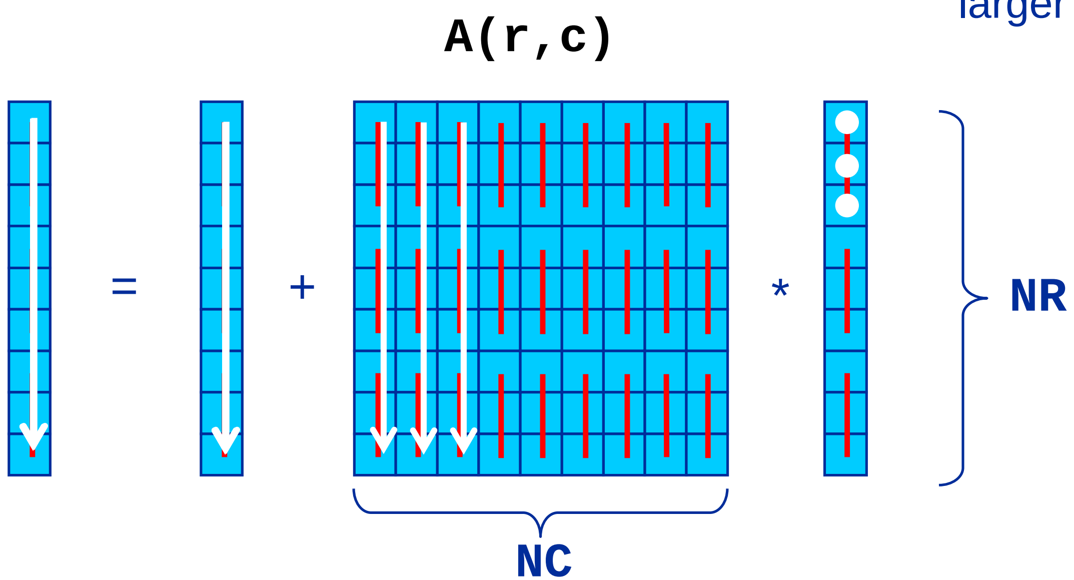
```
do c = 1 , NC
  tmp=x(c)
  do r = 1 , NR
     y(r)=y(r) + A(r,c)* tmp
  enddo
enddo
```

`tmp` stays in a register during inner loop

`A(:,:)` is loaded from memory – no data reuse

`y(:)` is loaded and stored in each outer iteration → for c>1 update `y(:)` in cache

`y(:)` may not fit in innermost cache → more traffic from lower level caches for larger **NR**



Analysis: Distinguish code balance in memory (B_C^{mem}) from code balance in relevant cache level(s) (B_C^{L3}, B_C^{L2},…)!

Code balance, reloaded!

- Code balance can be defined for any data path:

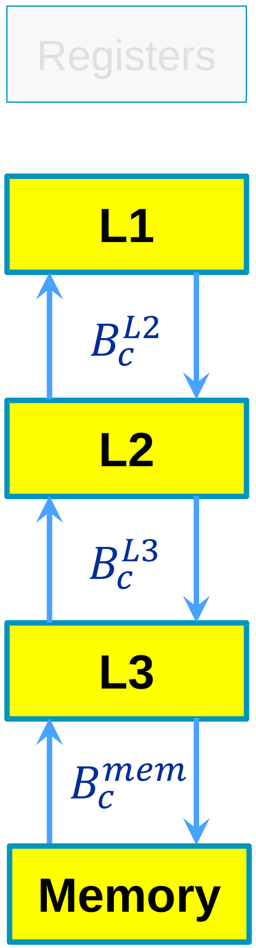
$$B_c^i = \frac{V_i}{W}$$

V_i = data volume over data path i
W = amount of work done with the data

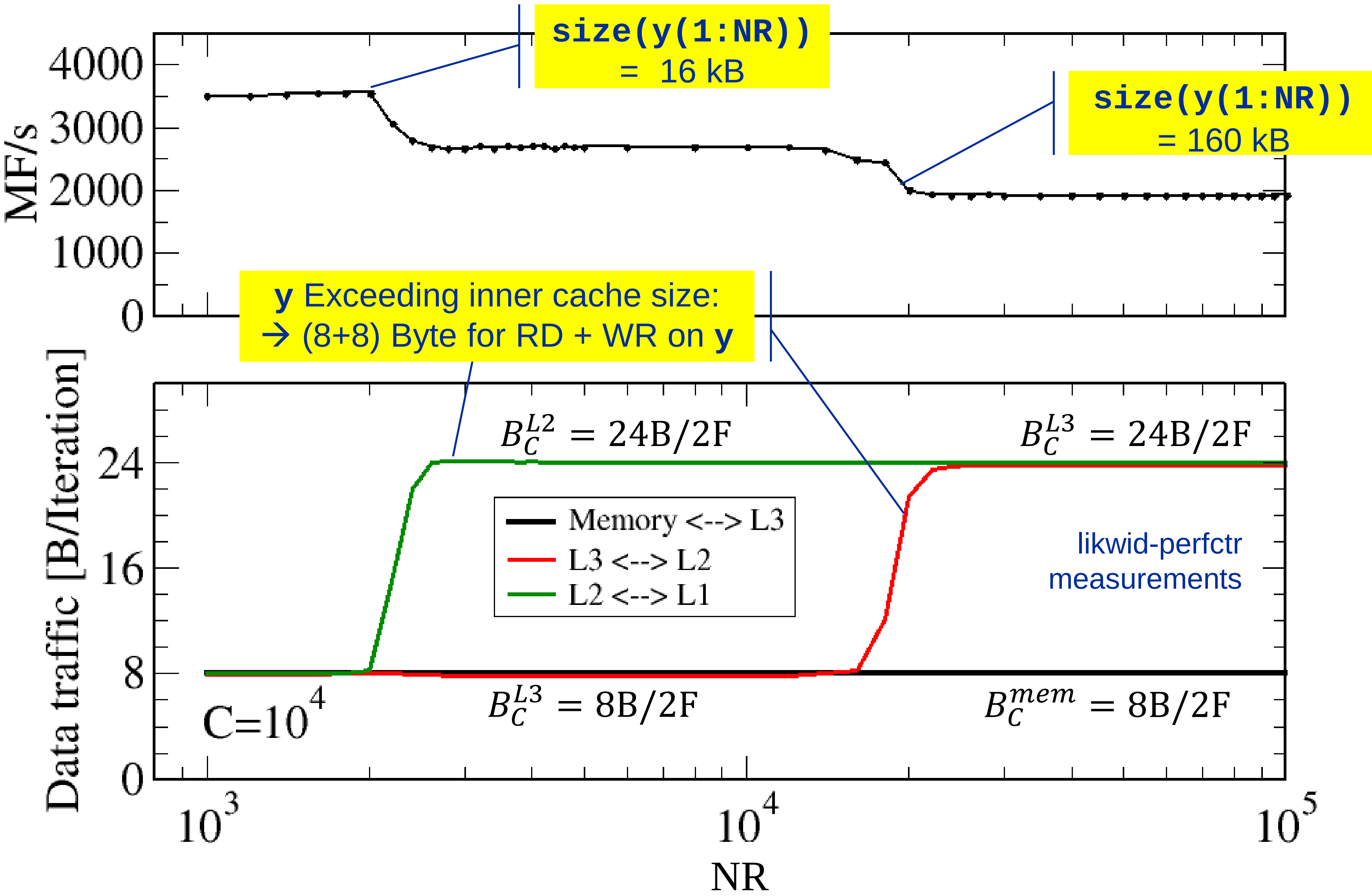
- *In principle*, the Roofline model can be expressed for those multiple bottlenecks:

$$P = \min\left(P_{\max}, \min_i \left[{b_S^i}/{B_c^i} \right]\right)$$

- However, the perfect overlap condition is invalid for the single-core cache hierarchy
  - But code balance is still useful for *qualitative* analysis…



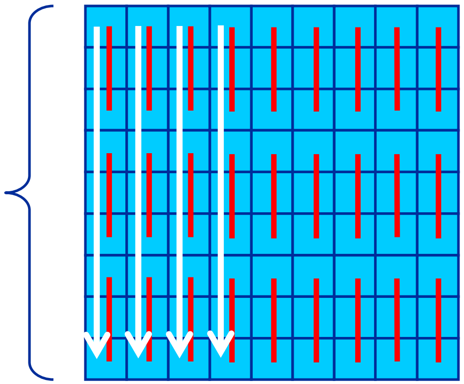
DMVM (DP) – Single core data traffic analysis



Reducing traffic by blocking

A(r,c)

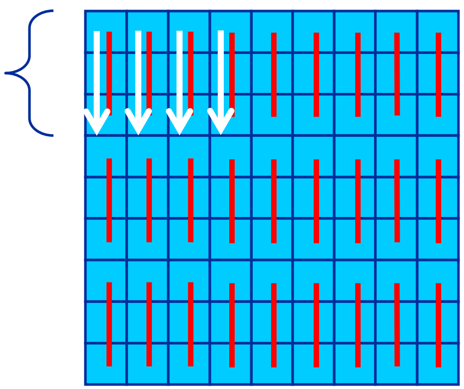
NR



```
do c = 1 , NC
  tmp=x(c)
  do r = 1 , NR
     y(r)=y(r) + A(r,c)* tmp
  enddo
enddo
```

y(:) may not fit into some cache → more traffic for lower level

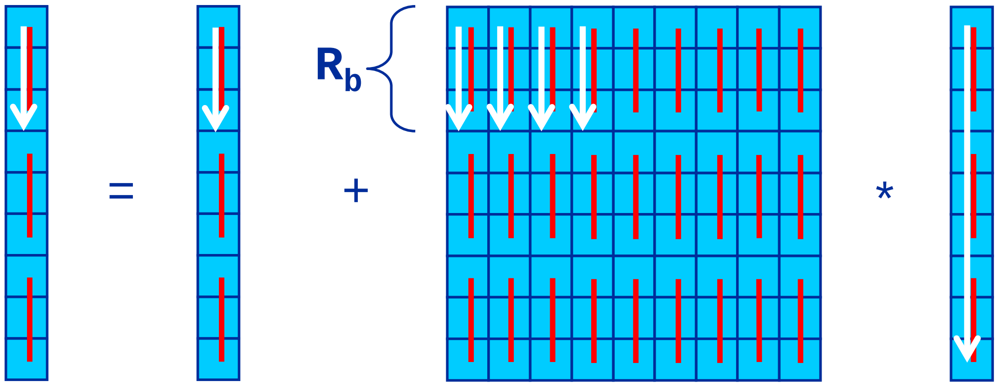
R_b



```
do rb = 1 , NR , Rb
 rbS = rb
 rbE = min((rb+Rb-1), NR)
 do c = 1 , NC
   do r = rbS , rbE
      y(r)=y(r) + A(r,c)*x(c)
   enddo
 enddo
enddo
```

y(rbS:rbE) may fit into some cache if R_b is small enough → traffic reduction

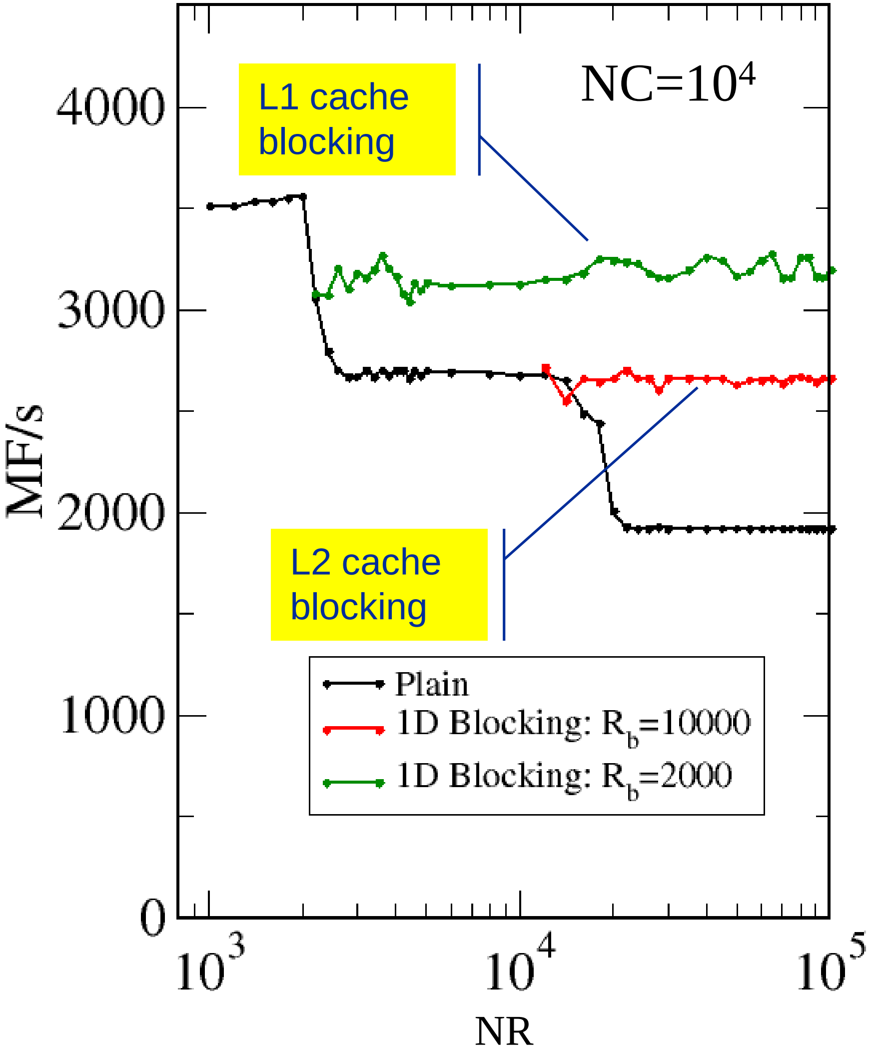
Reducing traffic by blocking



- LHS only updated once in some cache level if blocking is applied
- Price: RHS is loaded multiple times instead of once!
  - How often? → N_R / R_b times
  - RHS traffic: N_C x N_R / R_b
  - LHS traffic: 2 x N_R
  - Matrix: N_R x N_C

  Overall: $N_R \times \left(\frac{C}{R_b} + 2 + N_R\right) \approx N_R^2$ if $N_R, R_b \gg 1$ and $N_C = N_R$
- Without blocking: $N_R \times \left(\frac{N_C}{N_R} + 2N_C + N_R\right) \approx 3N_R^2$ if $N_R, R_b \gg 1$ and $N_C = N_R$

DMVM (DP) – Reducing traffic by inner loop blocking

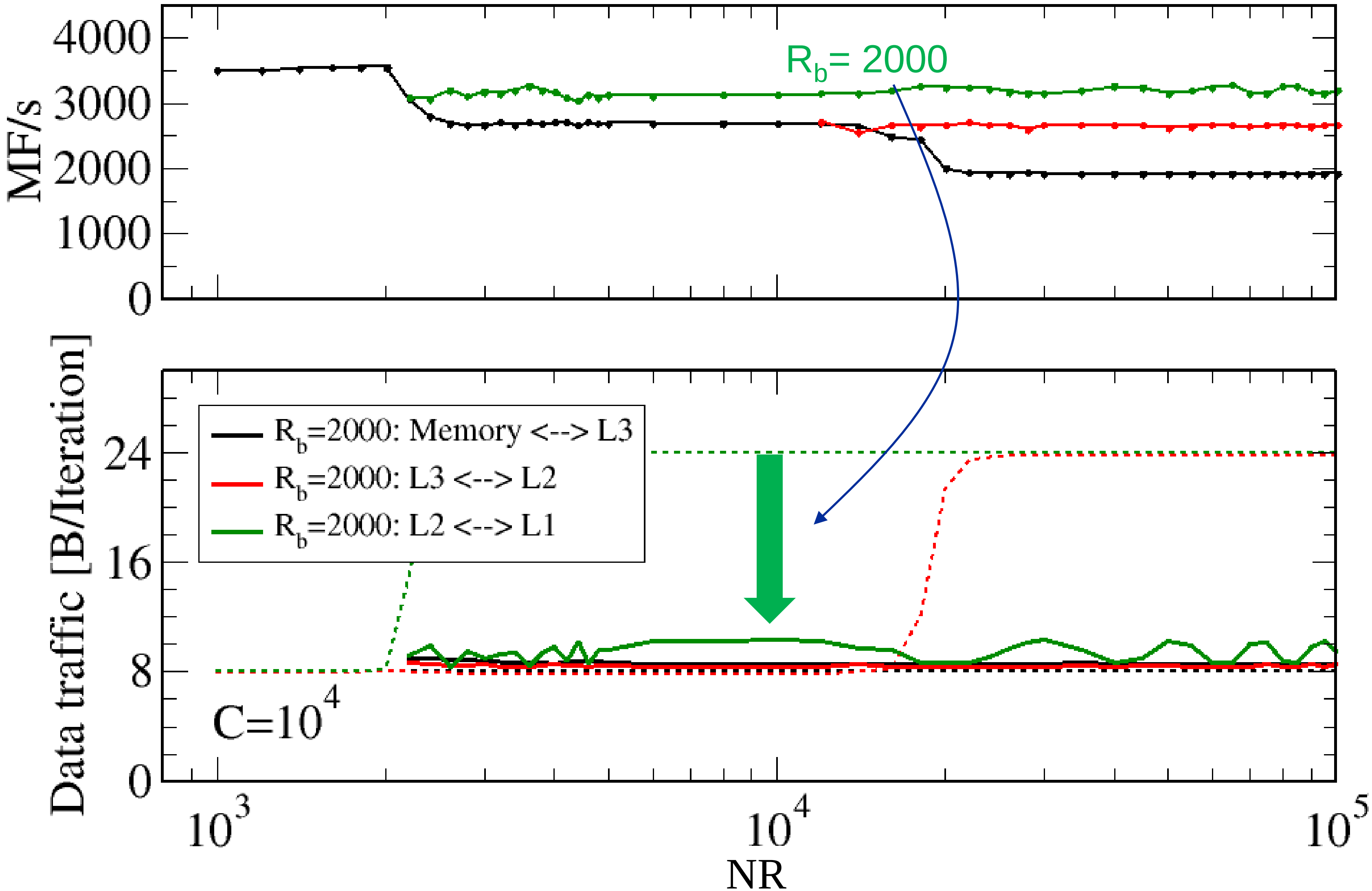


- "1D blocking" for inner loop
- Blocking factor $\mathbf{R_b}$ ←→ cache level

```
do rb = 1 , NR , Rb

  rbS = rb
  rbE = min((rb+Rb-1), NR)

  do c = 1 , NC
    do r = rbS , rbE
      y(r)=y(r) + A(r,c)*x(c)
    enddo
  enddo

enddo
```

→ Fully reuse subset of `y(rbS:rbE)` from L1/L2 cache

DMVM (DP) – Validation of blocking optimization



DMVM (DP) – OpenMP parallelization

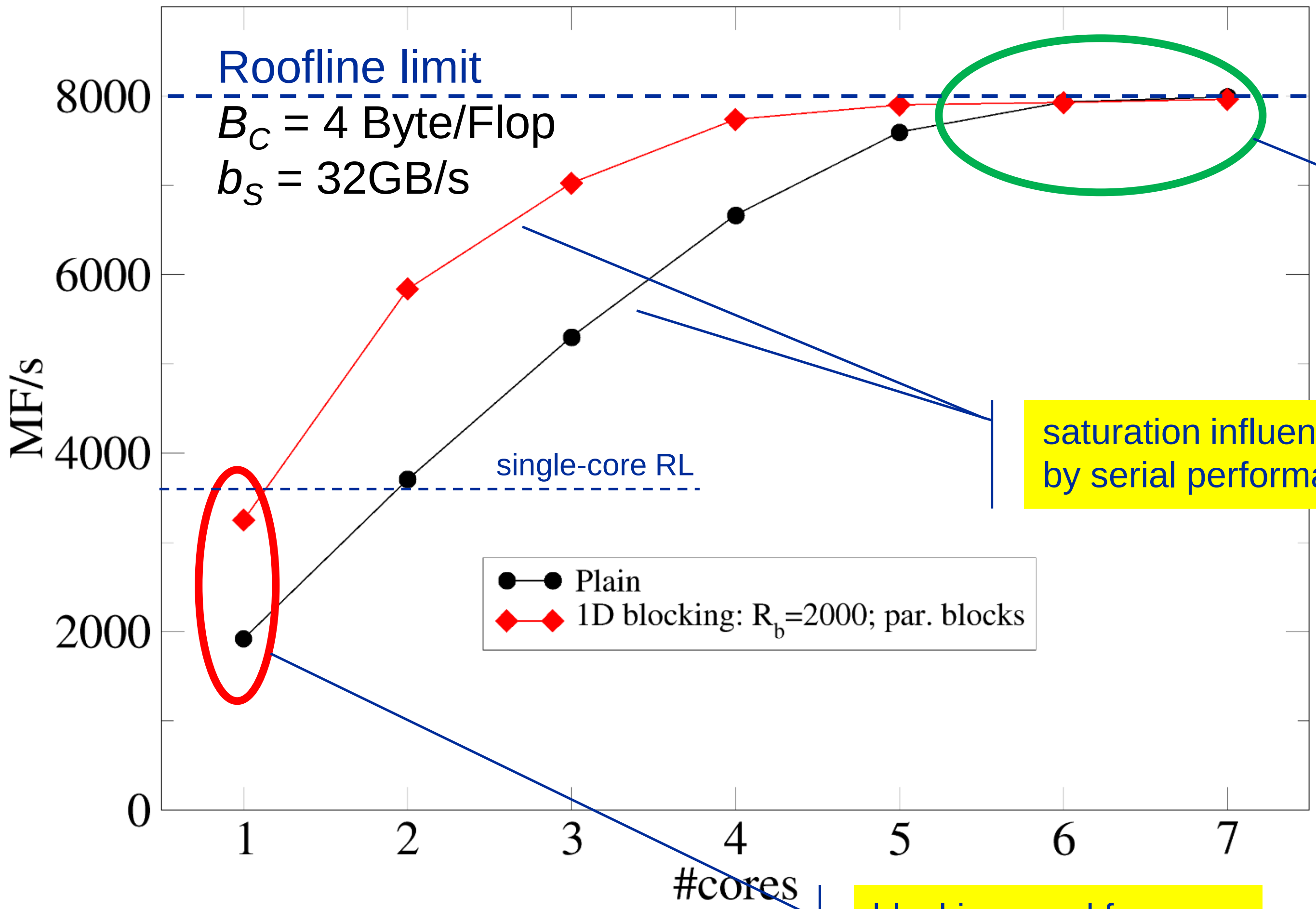
```
!$omp parallel do reduction(+:y)
do c = 1 , NC
  do r = 1 , NR
     y(r) = y(r) + A(r,c) * x(c)
enddo ; enddo
!$omp end parallel do
```

plain code

```
!$omp parallel do private(rbS,rbE)
do rb = 1 , NR , Rb
 rbS = rb
 rbE = min((rb+Rb-1), NR)
 do c = 1 , NC
   do r = rbS , rbE
      y(r) = y(r) + A(r,c) * x(c)
enddo ; enddo ; enddo
!$omp end parallel do
```

blocked code

DMVM (DP) – OpenMP parallelization & saturation



memory traffic unchanged
→ saturation unchanged!

saturation influenced by serial performance

blocking good for single thread performance (reduced in-cache traffic)

So, is blocking useless?
→ NO (see later)

Can we do anything to improve B_C^{mem}?
→ NO, not here

Intel Xeon E5 2695 v3 (Haswell-EP) CoD
2.3 GHz base clock speed, **b_S = 32 GB/s**

Conclusions from the dMVM example

- We have found the reasons for the breakdown of single-core performance with growing number of matrix rows
  - LHS vector fitting in different levels of the cache hierarchy
  - Validated theory by performance counter measurements
- Inner loop blocking was employed to improve code balance in L3 and/or L2
  - Validated by performance counter measurements
- Blocking led to better single-threaded performance
- Saturated performance unchanged (as predicted by Roofline)
  - Because the problem is still small enough to fit the LHS at least into the L3 cache